hep-th/0407009

Mirror Symmetry of Calabi-Yau Supermanifolds

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Abstract

We study super Landau-Ginzburg mirrors of the weighted projective superspace $\mathbf{WCP}^{3|2}$ which is a Calabi-Yau supermanifold and appeared in hep-th/0312171(Witten) in the topological B-model. One of them is an elliptic fibration over the complex plane whose coordinate is given in terms of two bosonic and two fermionic variables as well as Kahler parameter of $\mathbf{WCP}^{3|2}$. The other is some patch of a degree 3 Calabi-Yau hypersurface in $\mathbf{CP}^2$ fibered by the complex plane whose coordinate depends on both above four variables and Kahler parameter but its dependence behaves quite differently.

1 Introduction

From the equivalence between the perturbative expansion of $\mathcal{N}=4$ super Yang-Mills theory and the D-instanton expansion of a topological B-model on the Calabi-Yau *supermanifold* $\mathbf{CP}^{3|4}(1,1,1,1|1,1,1,1)$, the planar amplitudes of $\mathcal{N}=4$ super Yang-Mills theory are supported to holomorphic curves [1]. This target space has four bosonic homogeneous coordinates of weight 1 and four fermionic homogeneous coordinates of weight 1. The idea of Calabi-Yau supermanifold was found in [2] to resolve some issues in the mirror symmetry (for the review of this, see [3, 4]) by extending the space of bosonic Calabi-Yau manifold to the space of Calabi-Yau supermanifold. Moreover in [5], it was shown that certain Calabi-Yau space and Calabi-Yau superspace in the topological A-model are equivalent to each other.

The topological B-model on $\mathbf{CP}^{3|4}$ is mapped to A-model on $\mathbf{CP}^{3|4}$ through S-duality [6] and by mirror symmetry it was *conjectured* that the topological A-model on this Calabi-Yau supermanifold is equivalent to the topological B-model on a quadric in $\mathbf{CP}^{3|3}\times\mathbf{CP}^{3|3}$ [7]. The complex supermanifold $\mathbf{CP}^{3|3}$ has four bosonic homogeneous coordinates of weight 1 and three fermionic homogeneous coordinates of weight 1 and is not Calabi-Yau supermanifold. However, a quadric is a Calabi-Yau supermanifold. Moreover, it was *shown* that the topological A-model on the $\mathbf{CP}^{3|4}$ is mirror to the B-model on the quadric in $\mathbf{CP}^{3|3}\times\mathbf{CP}^{3|3}$ in the particular limit of Kahler parameter of $\mathbf{CP}^{3|4}$ [8]. Furthermore, the topological A-model on a quadric in $\mathbf{CP}^{3|3}\times\mathbf{CP}^{3|3}$ is mirror to the B-model on $\mathbf{CP}^{3|4}$ in a certain limit of one of the two Kahler parameters of the quadric and it has been checked the previous conjecture by [6] in [9].

In [1], it was suggested that a possible generalization of a target space can be done by giving a different weight to the fermionic coordinates without changing the bosonic manifold $\mathbf{CP}^3$. To construct Calabi-Yau supermanifold, the sum of bosonic weights equals that of fermionic weights. By allowing the fermionic weights to be positive and odd, one can write this supermanifold as a weighted projective superspace $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ uniquely. The central charge $\hat{c}$ for a conformal supersymmetric sigma model is given by $\hat{c}=3-2=1$ which is defined to be the net complex superdimension. The dimension of bosonic manifold is equal to 3 while the dimension of fermionic manifold is equal to 2.

In this note, we generalize the mirror geometry [3] from the linear sigma model description [10] in the context of bosonic Calabi-Yau manifold to the case in the above Calabi-Yau *supermanifold* $\mathbf{WCP}^{3|2}$. The idea of [8] is to introduce two additional fermions as well as a bosonic superfield when we take T-dualization for the phase of each fermion superfield. In section 2, starting with the super Landau-Ginzburg(LG) B-model mirror combined with the prescription of [8](the concept of T-duality for a fermion field), we compute some path integrations over dual bosonic and fermionic superfields. These manipulations are extremely 'formal' with target supermanifold in the sense that the field theory is nonunitary and usually does not have a good

vacuum. With the same spirit of [6], our starting point is to take A-model on $\mathbf{WCP}^{3|2}$ which is mapped to B-model on the same space by S-duality and study the B-model mirror of the topological A-model on $\mathbf{WCP}^{3|2}$. We use this conjecture and some formal manipulations to see where they lead to some hypersurface equation satisfied by mirror Calabi-Yau supermanifold and its mirror geometry. In section 3, after the summary of the paper, we make some remarks and future direction.

2 Mirror of $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$

The weighted projective superspace $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ has an extension of a linear sigma model [10] description in terms of four bosonic homogeneous coordinates $Z^I (I = 1,2,3,4)$ of weight 1 and two fermionic homogeneous coordinates ψ, χ of weights 1 and 3, respectively. Since the sum of bosonic weights equals the sum of fermionic weights, $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ is a Calabi-Yau supermanifold [1]. One can define a topological B-model with this target space [1, 11] and this supermanifold admits $\mathcal{N} = 1$ superconformal symmetry acting on both Z^I and ψ. The geometry of the linear sigma model with a given complexified Kahler class parameter t can be analyzed by solving the following D-term constraint

$$\sum_{I=1}^{4} |Z^I|^2 + |\psi|^2 + 3|\chi|^2 = \mathrm{Re}\, t$$

and dividing the gauge group $U(1)$. The vacuum structure depends on the signature of Kahler parameter $\mathrm{Re}\, t$.

Under the T-duality, the bosonic superfields Z^I of the linear sigma model are replaced by a dual cylinder-valued superfield Y_I, as usual, and the fermionic superfields ψ and χ are dualized to (X_1, η_1, χ_1) and (X_2, η_2, χ_2) respectively [8]. For each fermion field, there are a bosonic mirror as well as two additional fermions in order to preserve the central charge(that is, there is only one net fermionic dimension for the mirror). Here X_1 and X_2 are bosonic superfields while η_i and χ_i where $i = 1,2$ are *fermionic* superfields. Then the super Landau Ginzburg B-model mirror of $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ is given by the path integral for the holomorphic sector [8]

$$\int \prod_{I=1}^{4} dY_I \prod_{J=1}^{2} dX_J d\eta_J \mathrm{d}\chi_J \delta\left(\sum_{I=1}^{4} Y_I - X_1 - 3X_2 - t\right) \exp\left[\sum_{I=1}^{4} e^{-Y_I} + \sum_{J=1}^{2} e^{-X_J}\left(1 + \eta_J \chi_J\right)\right].$$

The superpotential of the mirror theory can be read off from the above exponent. The super LG model has 5 bosonic and 4 fermionic degrees of freedom(a mirror manifold has the same superdimension $5 - 4 = 1$ as the original one, by mirror symmetry). Note that there exists a single delta function constraint where $\mathrm{Re}\, Y_I = |Z^I|^2, \mathrm{Re}\, X_1 = -|\psi|^2$ and $\mathrm{Re}\, X_2 = -3|\chi|^2$.

We want to rewrite the above as a sigma model on a super Calabi-Yau hypersurface. To describe a mirror super Calabi-Yau interpretation for this LG, we should manipulate the integra-

tions over some superfields and successive superfield redefinitions. There exist two possibilities to carry out the integrations over the fermion superfields. In subsection 2.1, we first consider the case where the X_1 is given in terms of other variables and a complexified Kahler parameter t using the delta function constraint above. The final integral has 3 bosonic and 2 fermionic degrees of freedom. In subsection 2.2, we will come to the second case where we integrate out the X_2(as well as two fermionic superfields).

2.1 Integration over X_1

To describe a mirror super Calabi-Yau interpretation for this super LG, we integrate out some superfields with appropriate successive superfield redefinitions. In terms of Y_I and X_2, the delta function allows us to write X_1 as follows:

$$X_1 = -3X_2 - t + \sum_{I=1}^{4} Y_I. \tag{2.1}$$

We first integrate out the fermions η_1 and χ_1 and solve the delta function constraint for X_1. By computing the integrations over η_1, χ_1 and X_1, one gets the following B-model integral with 5 bosons and 2 fermions

$$\int \left(\prod_{I=1}^{4} dY_I\right) dX_2 d\eta_2 d\chi_2 \;\; e^{3X_2 - \sum_{I=1}^{4} Y_I} \exp\left[\sum_{I=1}^{4} e^{-Y_I} + e^{-X_2}\left(1+\eta_2\chi_2\right) + e^{t+3X_2-\sum_{I=1}^{4} Y_I}\right].$$

The nontrivial factors in the measure come from the integrating out the fermion superfields and we ignore an irrelevant normalization e^t. In the last term inside the exponent, we replaced X_1 with X_2, Y_I and t, according to the delta function constraint (2.1). Now let us introduce the new $\mathbf{C}$-valued bosonic superfields x_2 and y_I(which are good variables) as follows:

$$e^{-X_2} \equiv x_2, \qquad e^{-Y_I} \equiv y_I, \qquad I = 1, 2, 3, 4.$$

Then in terms of these new fields [1], the super LG model integral is given by

$$\int \left(\prod_{I=1}^{4} dy_I\right) \frac{dx_2}{x_2^4} d\eta_2 d\chi_2 \;\; \exp\left[\sum_{I=1}^{4} y_I + x_2\left(1+\eta_2\chi_2\right) + \frac{e^t y_1 y_2 y_3 y_4}{x_2^3}\right].$$

By redefining

$$\widetilde{y}_1 = y_1, \qquad \widetilde{y}_i \equiv \frac{y_i}{x_2}, \qquad i = 2, 3, 4$$

in order to make the super LG effective superpotential in the exponent to be a polynomial form(for Calabi-Yau supermanifold), one can reexpress it as

$$\int \left(\prod_{I=1}^{4} d\widetilde{y}_I\right) \left(\frac{dx_2}{x_2}\right) d\eta_2 d\chi_2 \;\; \exp\left[\widetilde{y}_1 + \sum_{i=2}^{4} x_2 \widetilde{y}_i + x_2\left(1+\eta_2\chi_2\right) + e^t \prod_{I=1}^{4} \widetilde{y}_I\right].$$

[1] One can also introduce the new variables $Y_i = \hat{Y}_i + Y_1$ where $i = 2, 3, 4$ and $X_2 = \hat{X}_2 + Y_1$ and in this case e^{-Y_1} plays the role of a Lagrange multiplier and the superpotential has an overall factor e^{-Y_1}.

Note that $\widetilde{y}_1$ is a Lagrange multiplier whose equation of motion is given by $1 + e^t \prod_{I=2}^4 \widetilde{y}_I = 0$. In order to absorb the nontrivial measure $1/x_2$ for x_2, let us introduce the two additional chiral bosonic superfields u and v taking values in $\mathbf{C}$ through a relation

$$\int dudve^{uvx_2} = \frac{1}{x_2}$$

enforcing x_2 to become a Lagrange multiplier due to the algebraic constraint. By performing the x_2 and $\widetilde{y}_1$-integrations, the LG period turns out to be

$$\int \left(\prod_{i=2}^{4} d\widetilde{y}_i\right) dudvd\eta_2 d\chi_2\ \delta\left(1 - uv + \sum_{i=2}^{4} \widetilde{y}_i + \eta_2\chi_2\right) \delta\left(1 + e^t \prod_{i=2}^{4} \widetilde{y}_i\right).$$

Finally the $\widetilde{y}_4$-integration(with the replacement of $t \to t + i\pi$) gives the following integral with 3 bosonic and 2 fermionic degrees of freedom(4 bosonic coordinates has one delta function constraint)

$$\int d\widetilde{y}_2 d\widetilde{y}_3 dudvd\eta_2 d\chi_2\ \delta\left(1 - uv + \widetilde{y}_2 + \widetilde{y}_3 + \frac{e^{-\widetilde{t}}}{\widetilde{y}_2\widetilde{y}_3} + \eta_2\chi_2\right). \tag{2.2}$$

The delta function inside the integral contains the information on the geometry of the mirror Calabi-Yau supermanifold.

Then the super LG mirror of $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$, by integrating out the dual fields corresponding to the fermion of weight 1, can be regarded as a super Calabi-Yau hypersurface characterized by

$$1 - uv + \widetilde{y}_2 + \widetilde{y}_3 + \frac{e^{-\widetilde{t}}}{\widetilde{y}_2\widetilde{y}_3} + \eta_2\chi_2 = 0 \tag{2.3}$$

where $\widetilde{y}_2$ and $\widetilde{y}_3$ take values in $\mathbf{C}^*$ and u and v are variables in $\mathbf{C}$. One can split this as

$$g(\widetilde{y}_2, \widetilde{y}_3) = uv - \eta_2\chi_2, \qquad \text{and} \qquad g(\widetilde{y}_2, \widetilde{y}_3) = 1 + \widetilde{y}_2 + \widetilde{y}_3 + \frac{e^{-\widetilde{t}}}{\widetilde{y}_2\widetilde{y}_3}.$$

The three critical points of $g(\widetilde{y}_2, \widetilde{y}_3)$ are given by $\widetilde{y}_i = \omega e^{-\frac{\widetilde{t}}{3}}$ where ω is a 3rd root of unity which yields the critical values in the $g(\widetilde{y}_2, \widetilde{y}_3)$ hypersurface on three points on a tiny circle of radius $|e^{-\frac{\widetilde{t}}{3}}|$ near $g(\widetilde{y}_2, \widetilde{y}_3) = 1$. This noncompact mirror Calabi-Yau has complex superdimension $(3|2)$ in toric supermanifold which is exactly the same as the dimension of the original $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$. Both supermanifolds have the same superdimension 1, as required by mirror symmetry because the superdimension determines the $\hat{c}$ in the current algebra.

At $\eta_2 = 0 = \chi_2$ patch, this hypersurface (2.3) in toric variety is exactly the noncompact bosonic Calabi-Yau threefold which is mirror to another noncompact bosonic Calabi-Yau threefold(there exists an obvious run-away direction of the superpotential), the line bundle

$\mathcal{O}(-3)$ over $\mathbf{CP}^2$ that is realized by a linear sigma model description in terms of a single $U(1)$ gauge theory with charges of the matter fields $(-3,1,1,1)$ without superpotential term [4] [2]. The field of -3 charge parametrizes the complex direction of the fiber and the fields with 1 charge correspond to span the base $\mathbf{CP}^2$. In this sense, the original Calabi-Yau supermanifold $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ contains noncompact bosonic Calabi-Yau threefold: the line bundle $\mathcal{O}(-3)$ over $\mathbf{CP}^2$.

The holomorphic volume form can be viewed as

$$\Omega = \frac{d\widetilde{y}_2 d\widetilde{y}_3 du dv d\eta_2 d\chi_2}{df}, \qquad f \equiv 1 - uv + \widetilde{y}_2 + \widetilde{y}_3 + \frac{e^{-\widetilde{t}}}{\widetilde{y}_2\widetilde{y}_3} + \eta_2\chi_2 = 0$$

and further v-integration on (2.2) gives the period of the holomorphic volume form $\int \Omega$. For better understanding the geometry of the mirror, by rescaling the fields [4],

$$\widetilde{y}_2 \to e^{-\frac{\widetilde{t}}{3}}\widetilde{y}_2, \quad \widetilde{y}_3 \to e^{-\frac{\widetilde{t}}{3}}\widetilde{y}_3, \quad u \to e^{-\frac{\widetilde{t}}{6}}u, \quad v \to e^{-\frac{\widetilde{t}}{6}}v, \quad \eta_2 \to e^{-\frac{\widetilde{t}}{6}}\eta, \quad \chi_2 \to e^{-\frac{\widetilde{t}}{6}}\chi$$

the defining equation (2.3) will be

$$\widetilde{y}_2^2\widetilde{y}_3 + \widetilde{y}_2\widetilde{y}_3^2 + \widetilde{y}_2\widetilde{y}_3 z + 1 = 0, \qquad z = e^{\frac{\widetilde{t}}{3}} - uv + \eta\chi,$$

and finally this [3] can be written in Weierstrass form

$$y^2 = x^3 + \left(\frac{z}{2}\right)^2 x^2 - \left(\frac{z}{2}\right) x + \frac{1}{4}, \qquad z - e^{\frac{\widetilde{t}}{3}} = -uv + \eta\chi. \tag{2.4}$$

The first equation defines an elliptic fibration over the complex plane with coordinate z and the second equation describes a $\mathbf{C}^*$-fibration over the (z,η,χ)-surface because for fixed values of (z,η,χ), from a relation $uv = \text{const}$, v can be written as $v = \text{const}/u$ and u can be any nonzero complex value. The general fiber is $\mathbf{C}^*$. At $z - e^{\frac{\widetilde{t}}{3}} = \eta\chi$, the $\mathbf{C}^*$-fibration degenerates when its nontrivial $\mathbf{S}^1$ shrinks. One can also interpret this as $\mathbf{C}^2$-fibration over (z,u)-surface since for fixed values of (z,u), one can express v in terms of η,χ that can be any complex values. Recall that u and v are bosonic superfields while η and χ are *fermionic* superfields.

Therefore, the super LG mirror of $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$, by integrating out the dual superfields corresponding to the fermion of weight 1, can be regarded as an elliptic fibration over z-plane with the second equation of (2.4).

[2]The LG superpotential of the mirror theory is given by $W = x_0 + x_1 + x_2 + e^{-t}\frac{x_0^3}{x_1 x_2}$ where $x_i = e^{-Y_i}$. Here Y_0 is the dual field to the charge -3 matter field and $Y_i (i=1,2,3)$ to the charge 1 matter fields. At $e^t = -27$, the singularity appears $\frac{\partial W}{\partial x_i} = 0$. In the last term of the superpotential we replaced Y_3 with $3Y_0 - Y_1 - Y_2 + t$ using a delta function constraint. Then it is easy to see [4] that the noncompact Calabi-Yau threefold defined by this LG superpotential is equivalent to another noncompact Calabi-Yau threefold defined by the above (2.3) with $\eta_2 = 0 = \chi_2$.

[3]One can introduce an extra variable x in order to make the equation homogenize: $\widetilde{y}_2^2\widetilde{y}_3 + \widetilde{y}_2\widetilde{y}_3^2 + x\widetilde{y}_2\widetilde{y}_3 z + x^3 = 0$. Then we take the coordinate transformation given by [4], $\widetilde{y}_2 = y - \left(\frac{z}{2}\right)x + \frac{1}{2}$ and $\widetilde{y}_3 = -y - \left(\frac{z}{2}\right)x + \frac{1}{2}$.

2.2 Integration over X_2

Let us consider the case where we integrate out X_2 instead of X_1. In this case, the delta function constraint will provide

$$X_2 = -\frac{1}{3}X_1 - \frac{t}{3} + \frac{1}{3}\sum_{I=1}^{4} Y_I.$$

By computing the integrations over X_2, η_2 and χ_2 as we have done before, one gets the following B-model integral with 5 bosons and 2 fermions

$$\int \prod_{I=1}^{4} dY_I dX_1 d\eta_1 d\chi_1 \; e^{\frac{1}{3}X_1 - \frac{1}{3}\sum_{I=1}^{4} Y_I} \exp\left[\sum_{I=1}^{4} e^{-Y_I} + e^{-X_1}(1+\eta_1\chi_1) + e^{\frac{1}{3}X_1 + \frac{t}{3} - \frac{1}{3}\sum_{I=1}^{4} Y_I}\right].$$

Now let us introduce the new $\mathbf{C}$-valued fields x_1 and y_I by realizing the measure factors:

$$e^{-\frac{X_1}{3}} \equiv x_1, \qquad e^{-\frac{Y_I}{3}} \equiv y_I.$$

Then in terms of these new fields the super LG model is given by

$$\int \left(\prod_{I=1}^{4} dy_I\right)\left(\frac{dx_1}{x_1^2}\right) d\eta_1 d\chi_1 \exp\left[\sum_{I=1}^{4} y_I^3 + x_1^3(1+\eta_1\chi_1) + \frac{e^{\frac{t}{3}} y_1 y_2 y_3 y_4}{x_1}\right].$$

By redefining

$$\widetilde{y}_1 \equiv \frac{y_1}{x_1}, \qquad \widetilde{y}_i = y_i, \qquad i = 2,3,4$$

in order to make the super LG superpotential in the exponent to be a polynomial form, one arrives at

$$\int \left(\prod_{I=1}^{4} d\widetilde{y}_I\right)\left(\frac{dx_1}{x_1}\right) d\eta_1 d\chi_1 \exp\left[x_1^3 \widetilde{y}_1^3 + \sum_{i=2}^{4} \widetilde{y}_i^3 + x_1^3(1+\eta_1\chi_1) + e^{\frac{t}{3}} \prod_{I=1}^{4} \widetilde{y}_I\right].$$

By using the relation $x_1^3 \equiv \widetilde{x}_1$ one can express this as

$$\int \left(\prod_{I=1}^{4} d\widetilde{y}_I\right)\left(\frac{d\widetilde{x}_1}{\widetilde{x}_1}\right) d\eta_1 d\chi_1 \exp\left[\widetilde{x}_1 \widetilde{y}_1^3 + \sum_{i=2}^{4} \widetilde{y}_i^3 + \widetilde{x}_1(1+\eta_1\chi_1) + e^{\frac{t}{3}} \prod_{I=1}^{4} \widetilde{y}_I\right].$$

In order to absorb the nontrivial measure $1/\widetilde{x}_1$ for $\widetilde{x}_1$ as we have done before, let us introduce the two additional chiral superfields u and v through $\int du dv e^{uv\widetilde{x}_1} = \frac{1}{\widetilde{x}_1}$ allowing $\widetilde{x}_1$ to become a Lagrange multiplier. Now $\widetilde{x}_1$-integration gives

$$\int \left(\prod_{I=1}^{4} d\widetilde{y}_I\right) du dv d\eta_1 d\chi_1 \; \delta\left(\widetilde{y}_1^3 + 1 + \eta_1\chi_1 - uv\right) \exp\left[\sum_{i=2}^{4} \widetilde{y}_i^3 + e^{\frac{t}{3}} \prod_{I=1}^{4} \widetilde{y}_I\right]. \tag{2.5}$$

Thus we have obtained a super submanifold defined by $\widetilde{y}_1^3+1+\eta_1\chi_1-uv=0$ and the expression (2.5) is identical to the period of super LG model of submanifold with superpotential

$$W=\sum_{i=2}^{4}\widetilde{y}_i^3+e^{\frac{t}{3}}\prod_{I=1}^{4}\widetilde{y}_I=\sum_{i=2}^{4}\widetilde{y}_i^3+\left(e^{\frac{t}{3}}\widetilde{y}_1\right)\prod_{i=2}^{4}\widetilde{y}_i. \tag{2.6}$$

How do we interpret this? For gauged linear sigma model [10] where a single $U(1)$ theory with charged matter fields with charges given by $(-n,1,1,1)$ in complex dimension 3, the $n=3$ case is conformal and corresponds to the $\mathcal{O}(-3)$ geometry over $\mathbf{CP}^2$ or its orbifold limit when $t\to-\infty$ given by $\mathbf{C}^3/\mathbf{Z}^3$. In this limit the superpotential has a simple form without the second term of (2.6) [12, 3]. In the present case, the dual field corresponding to the charge -3 is replaced by other dual fields through the delta function constraint, contrary to the previous case explained in the footnote 2 where one of the matter fields with charge 1 was replaced. Therefore when we integrated out a dual superfield X_1 corresponding to a fermionic superfield of weight 1 as in previous subsection 2.1, this led to the $\mathcal{O}(-3)$ geometry over $\mathbf{CP}^2$ with the replacement of charge 1 in a linear sigma model. On the other hand, when we integrate out a dual superfield X_2 corresponding to a fermionic superfield of weight 3 in this subsection, it produces the same Calabi-Yau threefold with the replacement of charge 3. The mirror description for linear sigma model is exactly given by the $\widetilde{y}_1=1$ patch of (2.6) modulo $\mathbf{Z}_3\times\mathbf{Z}_3$ which is the maximal group preserving all the monomials and $\mathbf{Z}_3$ acts as 3rd roots of unity on each field preserving all the monomials.

By the following redefinitions

$$\widetilde{y}_1=\hat{y}_1,\qquad \widetilde{y}_2^3=\hat{y}_2,\qquad \widetilde{y}_3=\widetilde{y}_2\hat{y}_3,\qquad \widetilde{y}_4=\widetilde{y}_2\hat{y}_4$$

and after $\hat{y}_2$-integration, the super LG model will lead to the following integral with 3 bosons and 2 fermions(there are two delta functions)

$$\int d\hat{y}_1d\hat{y}_3d\hat{y}_4dudvd\eta_1d\chi_1\,\delta\left(\hat{y}_1^3+1+\eta_1\chi_1-uv\right)\delta\left(1+\hat{y}_3^3+\hat{y}_4^3+e^{\frac{t}{3}}\hat{y}_1\hat{y}_3\hat{y}_4\right).$$

Then the mirror of $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ by integrating out the dual fields corresponding to the fermion of weight 3 can be regarded as a super Calabi-Yau hypersurface characterized by

$$\hat{y}_1^3+1+\eta_1\chi_1-uv=0,\qquad 1+\hat{y}_3^3+\hat{y}_4^3+e^{\frac{t}{3}}\hat{y}_1\hat{y}_3\hat{y}_4=0. \tag{2.7}$$

It is more convenient to introduce an extra variable $\hat{y}_0$ in the second equation of (2.7):

$$\hat{y}_0^3+\hat{y}_3^3+\hat{y}_4^3+\left(e^{\frac{t}{3}}\hat{y}_1\right)\hat{y}_0\hat{y}_3\hat{y}_4=0. \tag{2.8}$$

Then the equation (2.8) is invariant under

$$\hat{y}_0\to\gamma_0\hat{y}_0,\quad \hat{y}_3\to\gamma_3\hat{y}_3,\quad \hat{y}_4\to\gamma_4\hat{y}_4,\quad \hat{y}_1\to\gamma_1\hat{y}_1,\quad \gamma_i^3=1(i=0,3,4),\quad \gamma_0\gamma_1\gamma_3\gamma_4=1.$$

This is the equation for a Calabi-Yau hypersurface in $\mathbf{CP}^2$ with the parameter $e^{\frac{t}{3}}\hat{y}_1$: the degree 3 Calabi-Yau hypersurface in $\mathbf{CP}^2$ fibered over $\mathbf{C}$ [3]. The first equation of (2.7) is a $\mathbf{C}^*$-fibration over $(\hat{y}_1, \eta_1, \chi_1)$-surface because for fixed these values the relation $uv =$ const determines v in terms of nonzero complex value u which provides the fiber $\mathbf{C}^*$. The second equation of (2.7) is a $\hat{y}_0 = 1$ patch of (2.8). Of course, one can reduce to a single equation by susbstituting $\hat{y}_1$ from the first equation of (2.7) into the second equation of (2.7), but it is not clear whether this has any simple geometric interpretation or not.

Therefore, the super LG mirror of $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$, by integrating out the dual superfields corresponding to the fermion of weight 3, can be regarded as some patch of a degree 3 Calabi-Yau hypersurface in $\mathbf{CP}^2$ fibered by $\mathbf{C}$ with the first equation of (2.7).

3 Concluding remarks

In this paper, we have found that the super Landau-Ginzburg B-model mirrors of Calabi-Yau super manifold $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ can be described by a super Calabi-Yau hypersurface (2.3)(or (2.4)) or (2.7)(or (2.6)) depending on which dual superfields we take. As an bosonic submanifold, the original $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ contains a noncompact Calabi-Yau threefold, a line bundle $\mathcal{O}(-3)$ over $\mathbf{CP}^2$. The two dual *fermionic* superfields enter into either an elliptic fibration or some patch of cubic Calabi-Yau hypersurface in $\mathbf{CP}^2$ fibered over $\mathbf{C}$ differently because the original weighted projective superspace $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ possesses different weights in the *fermionic* superfields.

Although other bosonic complex manifold $\mathbf{CP}^{M-1}(M \neq 4)$ is not related to four-dimensional Minkowski spacetime by the Penrose transform [1], one also apply for the mirror of higher(only for bosonic dimension) dimensional weighted projective space $\mathbf{WCP}^{5|2}(1,1,\cdots,1|1,5)$ which has a linear sigma model in terms of six bosonic homogeneous coordinates Z^I of weight 1 and two fermionic coordinates ψ and χ of weights 1 and 5, respectively. This is also Calai-Yau supermanifold. Under the T-duality, the corrresponding super LG mirror of $\mathbf{WCP}^{5|2}(1,1,\cdots,1|1,5)$ is written as a path integral with eight bosonic and four fermionic dual variables with one single delta function and the superpotential has an extra two terms due to the extra two bosonic homogeneous coordinates, compared with $\mathbf{WCP}^{3|2}$. Following the procedures, i) computation for the integrations on η_1, χ_1 and X_1, ii) change to the right variables, iii) absorbing the nontrivial factor $1/x_2$, iv) integration for Lagrange multipliers, and v) using a delta function, as we have done in subsection 2.1, one gets

$$\int \left(\prod_{i=2}^{5} d\widetilde{y}_i\right) dudvd\eta_2 d\chi_2\, \delta\left(1 - uv + \sum_{i=2}^{5}\widetilde{y}_i + \frac{e^{-\widetilde{t}}}{\prod_{i=2}^{5}\widetilde{y}_i} + \eta_2\chi_2\right). \tag{3.1}$$

At $\eta_2 = 0 = \chi_2$ patch, the corresponding hypersurface in toric variety is exactly the noncompact

bosonic Calabi-Yau threefold which is equivalent to another noncompact bosonic Calabi-Yau threefold, the line bundle $\mathcal{O}(-5)$ over $\mathbf{CP}^4$. This is realized by a linear sigma model description in terms of a single $U(1)$ gauge theory with charges of the matter fields $(-5, 1, 1, 1, 1, 1)$ without superpotential term [13, 4]. The field of -5 charge corresponds to the fiber coordinate and the fields with 1 charge correspond to span the base $\mathbf{CP}^4$. For negative Re t, the space of fields is $\mathbf{C}^5/\mathbf{Z}_5$. The blow up of the origin in $\mathbf{C}^5/\mathbf{Z}_5$ is the line bundle $\mathcal{O}(-5)$ over $\mathbf{CP}^4$.

On the other hand, the X_2-integration gives other mirror. We can repeat the calculations by inserting the extra two dual bosonic variables and realizing the delta function constraint, as we have done in subsection 2.2. One arrives at

$$\int d\hat{y}_1 \left(\prod_{i=3}^{6} d\hat{y}_i\right) dudvd\eta_1 d\chi_1 \; \delta\left(\hat{y}_1^5 + 1 + \eta_1\chi_1 - uv\right) \delta\left(1 + \sum_{i=3}^{6} \hat{y}_i^5 + e^{\frac{t}{5}}\hat{y}_1 \prod_{i=3}^{6} \hat{y}_i\right). \tag{3.2}$$

The second delta function, when we introduce an extra variable $\hat{y}_0$, provides the equation for the Calabi-Yau hyperspace in $\mathbf{CP}^4$ with the parameter $e^{\frac{t}{5}}\hat{y}_1$. That is, a degree 5(well-known quintic) Calabi-Yau hypersurface in $\mathbf{CP}^4$ fibered over $\mathbf{C}$.

Therefore, we expect that from the Calabi-Yau supermanifold $\mathbf{WCP}^{M-1|2}(1, 1, \cdots, 1|1, M-1)$ which contains the line bundle $\mathcal{O}(-(M-1))$ over $\mathbf{CP}^{M-2}$, as a bosonic submanifold, there exist two mirror Calabi Yau supermanifolds for general M characterized by two equations similar to (3.1) and (3.2), by simple generalization(the indices for summation and product run from 1 to $M-2$).

For the weighted projective superspace with a *single* fermionic superfield, we expect, after some path integral, that the mirror we get corresponds to bosonic Calabi-Yau hypersurface in ordinary weighted projective space. One can consider the Calabi-Yau supermanifold $\mathbf{WCP}^{M-1|1}(1, 1, \cdots, 1|M)$ which has a linear sigma model description in terms of M bosonic homogeneous coordinates Z^I where $I = 1, 2, \cdots, M$ of weight 1 and one fermionic homogeneous coordinate ψ of weight M. Then the super LG B-model mirror of $\mathbf{WCP}^{M-1|1}(1, 1, \cdots, 1|M)$ is given by the path integral similarly [8]. The super LG model has $(M+1)$ bosonic and 2 fermionic degrees of freedom. Let us integrate out the fermions η and χ, solve the delta function constraint for X, and introduce the new $\mathbf{C}$-valued bosonic superfield y_I as follows:$e^{-\frac{Y_I}{M}} \equiv y_I, I = 1, 2, \cdots, M$. Then the super LG model is given by $\int \prod_{I=1}^{M} dy_I \;\exp\left[\sum_{I=1}^{M} y_I^M + e^{\frac{t}{M}} \prod_{I=1}^{M} y_I\right]$. This is exactly the bosonic Calabi-Yau manifold which is mirror to another bosonic Calabi-Yau manifold, that is realized by a linear sigma model description in terms of a single $U(1)$ gauge theory with charges of the matter fields $(-M, 1, 1, \cdots, 1)$. The field of $-M$ charge corresponds to the fiber coordinate and the fields with 1 charge correspond to span the base $\mathbf{CP}^{M-1}$ [3]. The super LG yields directly the periods of the bosonic Calabi-Yau manifold [5, 8]. Of course, the LG theory is given by the above superpotential modded out by $(\mathbf{Z}_M)^{M-1}$ which acts on each y_I by all M-th roots of unity preserving the product $\prod_{I=1}^{M} y_I$.

We would like to list several other interesting open problems for future directions.

• We have assumed the existence of a topological A-model on $\mathbf{WCP}^{3|2}$. It would be interesting to study how S-dual works [6], if there is, between A-model and B-model on the same Calabi-Yau supermanifold $\mathbf{WCP}^{3|2}$ and how the extra branes in the A-model and B-model affect this twistorial Calabi-Yau supermanifold.

• How the $\mathcal{N}=1$ superconformal field theory in four dimensions corresponding to a topological B-model on $\mathbf{WCP}^{3|2}(1,1,1,1|1,3)$ couples to conformal supergravity? Recently, it was found that the action of holomorphic Chern-Simons theory leads to some truncation of the self-dual $\mathcal{N}=4$ super Yang-Mills theory [11]. Then it would be interesting to find out the agreement of the twistor superfields in twistor-string theory with the physical states of $\mathcal{N}=4$ conformal supergravity in four dimensions [14], by some truncation. Or one can study it from the $\mathcal{N}=1$ conformal supergravity [15] in four dimensions directly. As already observed in [1], the topological B-model with target $\mathbf{WCP}^{3|2}$ should preserve $\mathcal{N}=1$ superconformal symmetry $SU(4|1)$ (while in $\mathcal{N}=1$ super Yang-Mills theory there is a conformal anomaly) and has additional symmetry which does not exist in $\mathcal{N}=1$ super Yang-Mills theory.

• In this note, we have started with a topological A-model on $\mathbf{WCP}^{3|2}$ and ended up with super LG B-model. How can we obtain super LG B-model mirror $\mathbf{WCP}^{3|2}$ from some unknown A-model? What is the correct A-model description? It would be interesting to see this by studying our super Calabi-Yau hypersurfaces carefully in some particular limit of t and to find out the correct linear sigma model description with appropriate charge assignments. Since for the Calabi-Yau condition, no $U(1)$ anomaly is allowed, the field content and $U(1)$ charge assignment should reflect this fact.

• There exist other kinds of Calabi-Yau supermanifolds by giving different weights for the fermionic superfields: $\mathbf{WCP}^{3|2}(1,1,1,1|2,2)$ and $\mathbf{WCP}^{3|2}(1,1,1,1|0,4)$ [11]. Now it is straightforward to apply our method to these Calabi-Yau supermanifolds. For the former, there exists a super submanifold defined by $1-uv+\widetilde{y}_3^2+\widetilde{y}_4^2+\eta_2\chi_2=0$ togther with the period of super LG model of submanifold with superpotential $W=\sum_{i=1}^{2}\widetilde{y}_i^2+e^{\frac{t}{2}}\prod_{i=1}^{4}\widetilde{y}_i$. Since the weights of the fermions are equal, it does not matter which one we integrate out. For the latter, we can obtain a super submanifold defined by $1+\eta_1\chi_1=0$ togther with the period of super LG model of submanifold with superpotential $W=\sum_{i=1}^{4}\widetilde{y}_i^4+e^{\frac{t}{4}}\prod_{i=1}^{4}\widetilde{y}_i$. There is only one mirror description because in the delta function constraint, there is no dependence on the dual bosonic superfield corresponding to fermion superfield of weight 0.

Acknowledgments

I would like to thank E. Buchbinder F. Cachazo, M. Fabinger, S. Hellerman, S. Prem

Kumar, K. Oh, P. Svrcek, C. Vafa and E. Witten for discussions. This research was supported by a grant in aid from the Monell Foundation through Institute for Advanced Study, by SBS Foundation, and by Korea Research Foundation Grant (KRF-2002-015- CS0006).

References

[1] E. Witten, "Perturbative gauge theory as a string theory in twistor space," [arXiv:hep-th/0312171].

[2] S. Sethi, "Supermanifolds, rigid manifolds and mirror symmetry," Nucl. Phys. B **430**, 31 (1994) [arXiv:hep-th/9404186].

[3] K. Hori and C. Vafa, "Mirror symmetry," [arXiv:hep-th/0002222].

[4] K. Hori, A. Iqbal and C. Vafa, "D-branes and mirror symmetry," [arXiv:hep-th/0005247].

[5] A. Schwarz, "Sigma models having supermanifolds as target spaces," Lett. Math. Phys. **38**, 91 (1996) [arXiv:hep-th/9506070].

[6] A. Neitzke and C. Vafa, "N = 2 strings and the twistorial Calabi-Yau," [arXiv:hep-th/0402128].

[7] E. Witten, "An Interpretation Of Classical Yang-Mills Theory," Phys. Lett. B **77**, 394 (1978).

[8] M. Aganagic and C. Vafa, "Mirror symmetry and supermanifolds," [arXiv:hep-th/0403192].

[9] S. P. Kumar and G. Policastro, "Strings in twistor superspace and mirror symmetry," [arXiv:hep-th/0405236].

[10] E. Witten, "Phases of N = 2 theories in two dimensions," Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[11] A. D. Popov and M. Wolf, "Topological B-Model on Weighted Projective Spaces and Self-Dual Models in Four Dimensions," [arXiv:hep-th/0406224].

[12] C. Vafa, "Mirror symmetry and closed string tachyon condensation," [arXiv:hep-th/0111051].

[13] T. m. Chiang and B. R. Greene, "Phases of mirror symmetry," [arXiv:hep-th/9509049].

[14] N. Berkovits and E. Witten, "Conformal supergravity in twistor-string theory," [arXiv:hep-th/0406051].

[15] E. S. Fradkin and A. A. Tseytlin, "Conformal Supergravity," Phys. Rept. **119**, 233 (1985).